

# STABLE GEOMETRIC PROPERTIES OF PLURIHARMONIC AND BIHOLOMORPHIC MAPPINGS, AND LANDAU-BLOCH'S THEOREM

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**ABSTRACT.** In this paper, we investigate some properties of pluriharmonic mappings defined in the unit ball. First, we discuss some geometric univalence criteria on pluriharmonic mappings, and then establish a Landau-Bloch theorem for a class of pluriharmonic mappings.

## 1. INTRODUCTION AND MAIN RESULTS

As usual,  $\mathbb{C}^n$  denotes the complex Euclidean space of  $n$  variables  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$  with the standard *Hermitian inner product*  $\langle z, w \rangle := \sum_{k=1}^n z_k \bar{w}_k$  and norm  $\|z\| := \langle z, z \rangle^{1/2}$ , where  $w = (w_1, \dots, w_n)$ , and  $\bar{w}_k$  ( $1 \leq k \leq n$ ) denotes the complex conjugate of  $w_k$  with  $\bar{w} = (\bar{w}_1, \dots, \bar{w}_n)$ . For  $a \in \mathbb{C}^n$ ,

$$\mathbb{B}^n(a, r) = \{z \in \mathbb{C}^n : \|z - a\| < r\}$$

denotes the (open) ball of radius  $r > 0$  with center  $a$  and

$$\partial \mathbb{B}^n(a, r) = \{z \in \mathbb{C}^n : \|z - a\| = r\}.$$

Also, we let  $\mathbb{B}^n(r) := \mathbb{B}^n(0, r)$ , and use  $\mathbb{B}^n$  to denote the unit ball  $\mathbb{B}^n(1)$ , and  $\mathbb{D} = \mathbb{B}^1$ .

Throughout,  $\mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$  denotes the set of all mappings  $f$  from  $\mathbb{B}^n$  into  $\mathbb{C}^n$  which are continuously differentiable as mappings into  $\mathbb{R}^{2n}$  with  $f = (f_1, \dots, f_n)$  and  $f_j(z) = u_j(z) + iv_j(z)$  ( $1 \leq j \leq n$ ), where  $u_j$  and  $v_j$  are real-valued functions from  $\mathbb{B}^n$  into  $\mathbb{R}$ .

For a complex-valued and differentiable function  $f$  from  $\mathbb{B}^n$  into  $\mathbb{C}$ , we introduce (see for instance [6, 7, 8])

$$\nabla f = \left( \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right) \quad \text{and} \quad \bar{\nabla} f = \left( \frac{\partial f}{\partial \bar{z}_1}, \dots, \frac{\partial f}{\partial \bar{z}_n} \right).$$

For  $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$ , we use  $J_f$  to denote the *real Jacobian matrix* of  $f$  (cf. [4]). Moreover, for each  $f = (f_1, \dots, f_n) \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$ , denote by

$$Df = (\nabla f_1, \dots, \nabla f_n)^T$$

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the matrix whose rows are the complex gradients  $\nabla f_1, \dots, \nabla f_n$ , and let

$$\overline{D}f = (\overline{\nabla}f_1, \dots, \overline{\nabla}f_n)^T,$$

where  $T$  means the matrix transpose.

For an  $n \times n$  complex matrix  $A$ , we introduce the *operator norm*

$$\|A\| = \sup_{z \neq 0} \frac{\|Az\|}{\|z\|} = \max \{ \|A\theta\| : \theta \in \partial\mathbb{B}^n \}.$$

We use  $L(\mathbb{C}^n, \mathbb{C}^n)$  to denote the space of continuous *linear operators* from  $\mathbb{C}^n$  into  $\mathbb{C}^n$  with the operator norm, and let  $I_n$  be the *identity operator* in  $L(\mathbb{C}^n, \mathbb{C}^n)$ .

It follows from [20, Theorem 4.4.9] that a real-valued function  $u$  defined on a simply connected domain  $G$  is pluriharmonic if and only if  $u$  is the real part of a holomorphic function on  $G$ . Clearly, a mapping  $f : \mathbb{B}^n \rightarrow \mathbb{C}$  is pluriharmonic if and only if  $f$  has a representation  $f = h + \overline{g}$ , where  $g$  and  $h$  are holomorphic. We refer to [4, 6, 13, 15, 20] for the definition and further details on pluriharmonic mappings.

A *vector-valued mapping*  $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$  is said to be pluriharmonic, if each of its component functions is a pluriharmonic mapping from  $\mathbb{B}^n$  into  $\mathbb{C}$ . We denote by  $\mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  the set of all *vector-valued pluriharmonic mappings* from  $\mathbb{B}^n$  into  $\mathbb{C}^n$ . Let  $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ , where  $h$  and  $g$  are holomorphic in  $\mathbb{B}^n$ . Then

$$\det J_f = \det \begin{pmatrix} Dh & \overline{Dg} \\ Dg & \overline{Dh} \end{pmatrix}$$

and if, in addition,  $h$  is locally biholomorphic, then one can easily get the formula

$$\det J_f = |\det Dh|^2 \det \left( I_n - Dg[Dh]^{-1} \overline{Dg[Dh]^{-1}} \right).$$

In the case of a *planar harmonic mapping*  $f = h + \overline{g}$ , we find that

$$\det J_f = |f_z|^2 - |\overline{f_z}|^2$$

and so,  $f$  is locally univalent and sense-preserving in  $\mathbb{D}$  if and only if  $|f_{\overline{z}}(z)| < |f_z(z)|$  in  $\mathbb{D}$ ; or equivalently if  $f_z(z) \neq 0$  and the dilatation  $\omega(z) = \overline{f_{\overline{z}}(z)}/f_z(z)$  is analytic in  $\mathbb{D}$  and has the property that  $|\omega(z)| < 1$  in  $\mathbb{D}$  (see [12, 16]). For  $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ , the condition  $\|Dg[Dh]^{-1}\| < 1$  is sufficient for  $\det J_f$  to be positive and hence for  $f$  to be sense-preserving. This is indeed a natural generalization of one-variable condition. (cf. [13]).

Throughout the discussion a diagonal matrix  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$  will be denoted for convenience by  $A = A(\lambda)$  with an understanding that the diagonal entries are  $\lambda_j$ ,  $j = 1, 2, \dots, n$ , i.e.  $(j, j)$ -th entry of the  $n \times n$  matrix  $A$  is  $\lambda_j$ .

**Definition 1.** Let  $f = h + \overline{g}$  be a pluriharmonic mapping from  $\mathbb{B}^n$  into  $\mathbb{C}^n$ , where  $h$  and  $g$  are holomorphic in  $\mathbb{B}^n$ . Then

- (1)  $f$  is called *stable pluriharmonic univalent* in  $\mathbb{B}^n$  if for every  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $\|A\| = 1$ , the mappings  $f_A = h + \overline{g}A$  are univalent in  $\mathbb{B}^n$ .
- (2)  $f$  is called *stable diagonal pluriharmonic univalent* in  $\mathbb{B}^n$  if for every diagonal matrix  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ , the mappings  $f_A = h + \overline{g}A$  are univalent in  $\mathbb{B}^n$  and  $A = A(\lambda)$  with  $|\lambda_j| = 1$  for each  $j \in \{1, 2, \dots, n\}$ .

The class of all stable pluriharmonic univalent mappings (resp. stable diagonal pluriharmonic univalent mappings) in  $\mathbb{B}^n$  is denoted by SPU (resp. SDPU) (cf. [14]).

Similarly, we have

**Definition 2.** Let  $F = h + g$  be a holomorphic mapping from  $\mathbb{B}^n$  into  $\mathbb{C}^n$ , where  $h$  and  $g$  are holomorphic in  $\mathbb{B}^n$ . Then we say that

- (1)  $F$  is *stable biholomorphic* in  $\mathbb{B}^n$  if for every  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $\|A\| = 1$ , the mappings  $h + gA$  are biholomorphic in  $\mathbb{B}^n$ .
- (2)  $F$  is *stable diagonal biholomorphic* in  $\mathbb{B}^n$  if for every diagonal matrix  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ , the mappings  $h + gA$  are biholomorphic, where  $A = A(\lambda)$  with  $|\lambda_j| = 1$  for each  $j \in \{1, 2, \dots, n\}$ .

The class of all stable biholomorphic mappings (resp. stable diagonal biholomorphic mappings) in  $\mathbb{B}^n$  is denoted by SBH (resp. SDBH).

Now, we state our first result.

**Theorem 1.** *The pluriharmonic mapping  $f = h + \bar{g}$  is in SPU (resp. SDPU) if and only if the holomorphic mapping  $F = h + g$  is in SBH (resp. SDBH). Moreover, if  $f = h + \bar{g}$  is in SPU (resp. SDPU) with  $\|Dg[Dh]^{-1}\| < 1$ , where  $h$  is locally biholomorphic and  $g$  is holomorphic in  $\mathbb{B}^n$ . Then  $h$  is biholomorphic in  $\mathbb{B}^n$ .*

If we give a stronger assumption, then we can prove a more general class of holomorphic mappings  $F_A = h + gA$  are biholomorphic in  $\mathbb{B}^n$ , where  $A$  is a diagonal matrix with  $\|A\| \leq 1$ . The result is as follows.

**Proposition 1.1.** *Let  $f = h + \bar{g}$  belong to SDPU with  $\|Dg[Dh]^{-1}\| < 1$ , where  $h$  is locally biholomorphic and  $g$  is holomorphic in  $\mathbb{B}^n$ . Then for every  $A = A(\lambda)$  with  $|\lambda_j| \leq 1$  for each  $j \in \{1, 2, \dots, n\}$ , the mappings  $F_A = h + gA$  are biholomorphic in  $\mathbb{B}^n$ .*

We remark that Proposition 1.1 is a generalization of [14, Theorem 7.1].

A domain  $D \subset \mathbb{C}^n$  is said to be *M-linearly connected* if there exists a positive constant  $M < \infty$  such that any two points  $w_1, w_2 \in D$  are joined by a path  $\gamma \subset D$  with

$$\ell(\gamma) \leq M\|w_1 - w_2\|,$$

where  $\ell(\gamma) = \inf \left\{ \int_{\gamma} \|dz\| : \gamma \subset D \right\}$ . It is not difficult to verify that a 1-linearly connected domain is convex. For extensive discussions on linearly connected domains, see [1, 5, 9, 10, 19].

In [10], the authors discussed the relationship between linear connectivity of the images of  $\mathbb{D}$  under the planar harmonic mappings  $f = h + \bar{g}$  and under their corresponding holomorphic counterparts  $h$ , where  $h$  and  $g$  are holomorphic in  $\mathbb{D}$ . In [11, Theorem 5.3], Clunie and Sheil-Small established an effective and beautiful method of constructing sense-preserving univalent harmonic mappings defined on the unit disk, which is popularly called shear construction. The following result is a generalizations of the shearing theorem of Clunie and Sheil-Small.

**Theorem 2.** *Let  $f = h + \bar{g}$  be a pluriharmonic mapping from  $\mathbb{B}^n$  into  $\mathbb{C}^n$  and  $F = h - g$ , where  $h$  is locally biholomorphic and  $g$  is holomorphic in  $\mathbb{B}^n$ .*

(I) *If  $F$  is biholomorphic and  $\Omega = F(\mathbb{B}^n)$  is  $M$ -linearly connected with*

$$\|Dg(z)[Dh(z)]^{-1}\| \leq C < \frac{1}{2M+1} \quad \text{for } z \in \mathbb{B}^n,$$

*then for every  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $\|A\| \leq 1$ , the mapping  $f_A = h + \bar{g}A$  is univalent and  $f_A(\mathbb{B}^n)$  is  $M'$ -linearly connected, where  $M' = \frac{M(1+C)}{1-(1+2M)C}$ . In particular,  $h + g$  is in SBH.*

(II) *If  $f$  is univalent and  $\Omega = f(\mathbb{B}^n)$  is  $M$ -linearly connected with*

$$\|Dg(z)[Dh(z)]^{-1}\| \leq C < \frac{1}{2M+1} \quad \text{for } z \in \mathbb{B}^n,$$

*then for every  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $\|A\| \leq 1$ , the mapping  $F_A = h - gA$  is univalent and  $F_A(\mathbb{B}^n)$  is  $M'$ -linearly connected, where  $M' = \frac{M(1+C)}{1-(1+2M)C}$ . In particular,  $h + \bar{g}$  belongs to SPU.*

For  $r \in (0, 1)$  and a mapping  $f = h + \bar{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  with  $\|Dg[Dh]^{-1}\| < 1$ , the generalized volume function  $V_f(r)$  of  $f$  is defined by

$$V_f(r) = \int_{\mathbb{B}^n(r)} \|Dh(z)\|^{2n} (1 - \|Dg(z)[Dh(z)]^{-1}\|^2)^n dV(z),$$

where  $h$  is locally biholomorphic and  $g$  is holomorphic in  $\mathbb{B}^n$ , and  $dV$  denotes the normalized Lebesgue volume measure on  $\mathbb{B}^n$ . We denote by  $\mathcal{PH}(V)$ , the class of all mappings  $f = h + \bar{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  with the finiteness condition

$$V := \sup_{0 < r < 1} V_f(r) < \infty.$$

We remark that if  $n = 1$ , then

$$V_f(r) = \int_{\mathbb{D}_r} \det J_f(z) dA(z)$$

is the *area function* of the planar harmonic mapping  $f$  defined in  $\mathbb{D}$ , where  $dA$  denotes the normalized Lebesgue area measure on  $\mathbb{D}$ .

For a holomorphic mapping  $f : \mathbb{B}^n \rightarrow \mathbb{C}^n$ ,  $\mathbb{B}^n(a, r)$  is called a *schlicht ball* of  $f$  if there is a subregion  $\Omega \subset \mathbb{B}^n$  such that  $f$  maps  $\Omega$  biholomorphically onto  $\mathbb{B}^n(a, r)$ . We denote by  $B_f$  the least upper bound of radii of all schlicht balls contained in  $f(\mathbb{B}^n)$  and call this the *Landau-Bloch radius* of  $f$ . The classical theorem of Landau-Bloch for holomorphic functions in the unit disk fails to extend to general holomorphic mappings in the ball of  $\mathbb{C}^n$  (see [21, 22]). However, in 1946, Bochner [2] proved that Landau-Bloch's theorem does hold for a class of real harmonic quasiregular mappings. For the extensive discussions on this topic, see [3, 4, 6, 7, 8, 17, 18]. Our next aim is to establish a Landau-Bloch Theorem for  $f \in \mathcal{PH}(V)$ .

**Theorem 3.** *Let  $f = h + \bar{g} \in \mathcal{PH}(V)$ , where  $V$  is a positive constant,  $f(0) = 0$ ,  $Dg(0) = 0$  and  $|\det J_f(0)| = \alpha$  for some positive constant  $\alpha$ ,  $0 < \alpha \leq \frac{8V\psi_0^n}{\pi}$  with*

$\psi_0 = (11 + 5\sqrt{5})/2 \approx 11.09017$ ,  $h$  is locally biholomorphic and  $g$  is holomorphic. Then  $f$  is univalent in  $\mathbb{B}^n(R_u)$ , and  $f(\mathbb{B}^n(R_u))$  covers the ball  $\mathbb{B}^n(R_c)$ , where

$$R_u \geq \frac{\alpha\pi(\sqrt{5}-1)(3-2\sqrt{2})}{8V\psi_0^n} \quad \text{and} \quad R_c \geq \frac{\alpha^2\pi(\sqrt{5}-1)(3-2\sqrt{2})}{16V^{\frac{4n-1}{2n}}\psi_0^{\frac{4n-1}{2}}}.$$

In fact, the following example will show that there is no Landau-Bloch theorem for a class of pluriharmonic mappings with finite volume.

**Example 1.1.** For  $k \in \{1, 2, \dots\}$  and  $z = (z_1, \dots, z_n) \in \mathbb{B}^n$ , let

$$f_k(z) = (kz_1, z_2/k, z_3, \dots, z_n).$$

It is not difficult to see that  $\det J_{f_k}(0) - 1 = |f_k(0)| = 0$  and the volume of  $f_k(\mathbb{B}^n)$  is

$$\int_{\mathbb{B}^n} |\det J_{f_k}(z)| dV(z) < \infty.$$

But there is no absolute constant  $s > 0$  such that  $\mathbb{B}^n(s)$  belongs to  $f_k(\mathbb{B}^n)$  for all  $k \in \{1, 2, \dots\}$ .

The following result provides a relationship between the real volume and the generalized volume on a pluriharmonic mapping  $f$  defined in  $\mathbb{B}^n$ .

**Theorem 4.** Let  $f = h + \bar{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  with  $\|Dg[Dh]^{-1}\| < 1$ , where  $h$  is locally biholomorphic and  $g$  is holomorphic in  $\mathbb{B}^n$ . Then for  $r \in (0, 1)$ ,

$$\int_{\mathbb{B}^n(r)} |\det J_f(z)| dV(z) \leq K_r^n V_f(r),$$

where  $K_r = \frac{1+r}{1-r}$ .

We remark that Chen and Gauthier proved Landau-Bloch type Theorems for bounded planar harmonic mappings (resp. pluriharmonic mappings) (see [4, Theorem 3] (resp. [4, Theorem 5])). In fact, the volume of all bounded harmonic mappings (resp. pluriharmonic mappings) is finite. By Theorem 4, we see that the condition for functions with finitely generalized volume in Theorem 3 is weaker than the bounded functions condition in [4, Theorems 3 and 5]. In this sense, Theorem 3 generalizes [4, Theorems 3 and 5].

The proofs of Theorems 1.1 and 2 will be presented in Section 2, and the proof of Theorems 3 and 4 will be given in Section 3.

## 2. UNIVALENCE CRITERIA ON PLURIHARMONIC MAPPINGS

**Proof of Theorem 1.** We begin to prove the necessity of the first part. Let  $f = h + \bar{g}$  belong to SPU. Then, for each  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $\|A\| = 1$ , the mappings  $f_A = h + \bar{g}A$  are univalent in  $\mathbb{B}^n$ , where  $h = (h_1, \dots, h_n)$  and  $g = (g_1, \dots, g_n)$  are holomorphic in  $\mathbb{B}^n$ . Suppose on the contrary that  $F = h + g \notin \text{SBH}$ . Then there exists an  $A_0 \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $\|A_0\| = 1$  such that  $F_{A_0} = h + gA_0$  is not biholomorphic in  $\mathbb{B}^n$ . This means that there are two distinct points  $z_1, z_2 \in \mathbb{B}^n$  such that

$$(2.1) \quad F_{A_0}(z_1) = F_{A_0}(z_2), \quad \text{i.e.} \quad h(z_1) - h(z_2) = (g(z_2) - g(z_1))A_0.$$

We divide the rest of the arguments into two cases.

**Case 1.** If  $h(z_1) = h(z_2)$ , then  $g(z_1) = g(z_2)$ . This is a contradiction with the assumption.

**Case 2.** If  $h(z_1) \neq h(z_2)$ , then for  $j \in \{1, 2, \dots, n\}$ , we may let  $\theta_j = \arg(h_j(z_1) - h_j(z_2))$  so that (2.1) reduces to

$$(h(z_1) - h(z_2))B = (g(z_2) - g(z_1))A_0B,$$

which gives

$$(h(z_1) - h(z_2))B = \overline{(h(z_1) - h(z_2))} \overline{B} = \overline{(g(z_2) - g(z_1))} \overline{A_0} \overline{B},$$

or equivalently,

$$h(z_1) - h(z_2) = \overline{(g(z_2) - g(z_1))} \overline{A_0} \overline{B}^2,$$

where  $B = B(\lambda)$  is the diagonal matrix with  $\lambda_j = e^{-i\theta_j}$  for  $j = 1, 2, \dots, n$ . This contradicts the univalence of  $f_A = h + \overline{g}A$  for  $A = \overline{A_0} \overline{B}^2$ , which proves the necessity for  $f \in \text{SPU}$ . The necessity for  $f \in \text{SDPU}$  is the same, with  $A$  diagonal.

By using similar reasoning as in the proof of the necessity, we can get the proof of the sufficiency of the first part.

Now we begin to prove the second part of Theorem 1. We prove the second part with a method of contradiction. We suppose that  $h$  is not univalent in  $\mathbb{B}^n$ . Then there are two distinct points  $z_1, z_2 \in \mathbb{B}^n$  such that  $h(z_1) = h(z_2)$ . Without loss of generality, we assume that  $z_1 = h(z_1) = 0$ . In fact, we just take

$$F(z) = (h(\phi(z)) - h(z_1)) + \overline{(g(\phi(z)) - g(z_1))},$$

where  $\phi$  is the automorphism of  $\mathbb{B}^n$  such that  $\phi(0) = z_1$ .

Therefore, we can assume that  $f = h + \overline{g}$  is in  $\text{SPU}$  (resp.  $\text{SDPU}$ ) with the normalized conditions  $h(0) = g(0) = 0$ . By the conclusion of the first part of Theorem 1, we obtain that  $h + g$  belongs to  $\text{SBH}$  and therefore, it follows that for each  $z \in \mathbb{B}^n \setminus \{0\}$  either  $\|h(z)\| < \|g(z)\|$  or  $\|g(z)\| < \|h(z)\|$ . Now,  $h(z_2) = 0$  so  $\|h(z_2)\| < \|g(z_2)\|$ . Then by the continuity of the function  $\|h(z)\| - \|g(z)\|$ , we conclude that  $\|h(z)\| < \|g(z)\|$  in  $\mathbb{B}^n \setminus \{0\}$ .

Since  $h$  is locally biholomorphic, we see that there is a sequence of points in  $\{Z_n\}_{n \geq 1}$  in  $\mathbb{B}^n \setminus \{0\}$  such that  $Z_n \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,

$$\frac{\|h(Z_n)\|}{\|Z_n\|} = \left\| \frac{h(Z_n) - h(0)}{\|Z_n - 0\|} \right\| < \left\| \frac{g(Z_n) - g(0)}{\|Z_n - 0\|} \right\| = \frac{\|g(Z_n)\|}{\|Z_n\|} \quad \text{for } n \geq 1,$$

and therefore, we obtain that

$$(2.2) \quad \|Dh(0)\| \leq \|Dg(0)\|.$$

On the other hand, by the assumption, we have

$$\|Dg(0)\| = \|Dg(0)[Dh(0)]^{-1}Dh(0)\| \leq \|Dg(0)[Dh(0)]^{-1}\| \|Dh(0)\| < \|Dh(0)\|$$

which contradicts the inequality (2.2). The proof of this theorem is complete.  $\square$

**Proof of Proposition 1.1.** By the assumption of Proposition 1.1 and Theorem 1, we obtain that the mappings  $F_A = h + gA$  are biholomorphic for every diagonal matrix  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ , where  $A = A(\lambda)$  with  $|\lambda_j| = 1$  for each  $j = 1, 2, \dots, n$ .

Next we prove that for every diagonal matrix  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ , where  $A = A(\lambda)$  with  $|\lambda_j| < 1$  for some  $j \in \{1, 2, \dots, n\}$ , the mappings  $F_A = h + gA$  are also biholomorphic. Without loss of generality, we assume that diagonal matrix  $A(\lambda) \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $|\lambda_j| < 1$  for each  $j = 1, 2, \dots, n$ . In other words, we will prove that if  $f = h + \bar{g}$  belongs to SDPU and the diagonal matrix  $A = A(\lambda)$  is with  $|\lambda_j| < 1$  for each  $j = 1, 2, \dots, n$ , then we have  $T_A(f) = f + \bar{f}A \in \text{SDPU}$ .

By simple calculations, we have

$$T_A(f) = h + gA + \overline{g + h\bar{A}}.$$

By Theorem 1, in order to prove  $T_A(f) = f + \bar{f}A$  is in SDPU, we only need to prove that for every diagonal matrix of the form  $D = D(\lambda)$  with diagonal entries  $\lambda_j = e^{i\theta_j}$  ( $j = 1, 2, \dots, n$ ), the mappings  $h + gA + (g + h\bar{A})D$  are biholomorphic in  $\mathbb{B}^n$ , where  $\theta_j \in [0, 2\pi)$  for each  $j = 1, 2, \dots, n$ . We may rewrite

$$\begin{aligned} h + gA + (g + h\bar{A})D &= g(A + D) + h(I_n + \bar{A}D) \\ &= [h + g(A + D)(I_n + \bar{A}D)^{-1}](I_n + \bar{A}D). \end{aligned}$$

Then it is a simple exercise to see that the matrix  $C = (A + D)(I_n + \bar{A}D)^{-1}$  is a diagonal matrix with diagonal entries

$$\varphi_j(\lambda_j) = \frac{\lambda_j + e^{i\theta_j}}{1 + \bar{\lambda}_j e^{i\theta_j}} \quad \text{for } j = 1, 2, \dots, n.$$

Since  $|\varphi_j(\lambda_j)| = 1$  for each  $j = 1, 2, \dots, n$ , we conclude that

$$\|C\| = \|(A + D)(I_n + \bar{A}D)^{-1}\| = 1$$

and that the mapping  $h + gA + (g + h\bar{A})D$  belongs to SDBH so that  $T_A(f) = f + \bar{f}A$  is in SDPU. It follows from Theorem 1 that the holomorphic part  $h + gA$  of  $T_A(f)$  is biholomorphic. The proof of this proposition is complete.  $\square$

The following lemma plays a key role in the proofs of Theorem 2.

**Lemma A.** *Let  $A$  be an  $n \times n$  complex matrix with  $\|A\| < 1$ . Then  $I_n \pm A$  are nonsingular matrixes and  $\|(I_n \pm A)^{-1}\| \leq 1/(1 - \|A\|)$ .*

Lemma A may be referred to as the Neumann expansion theorem, and so the proof is omitted here.

**Proof of Theorem 2.** We first prove part (I). Let  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $\|A\| \leq 1$  and consider the mapping  $f_A = h + \bar{g}A$ . Using the hypotheses, we first show that  $f_A(\mathbb{B}^n)$  is linearly connected. Define  $\Omega = F(\mathbb{B}^n)$ , where  $F = h - g$ , and

$$H(w) = f_A(F^{-1}(w)) = w + g(F^{-1}(w)) + \overline{g(F^{-1}(w))}A \quad \text{for } w \in \Omega.$$

Clearly

$$DF = Dh - Dg = (I_n - Dg[Dh]^{-1})Dh,$$



and therefore, we see that

$$[DF]^{-1} = [Dh]^{-1} (I_n - Dg[Dh]^{-1})^{-1}$$

which gives

$$(2.3) \quad DH = I_n + Dg[DF]^{-1} = I_n + Dg[Dh]^{-1} (I_n - Dg[Dh]^{-1})^{-1}$$

and

$$(2.4) \quad \overline{DH} = \overline{Dg[DF]^{-1}A} = \overline{Dg[Dh]^{-1}(I_n - Dg[Dh]^{-1})^{-1}A}.$$

For any two distinct points  $w_1, w_2 \in \Omega$ , by hypothesis, there is a curve  $\gamma \subset \Omega$  joining  $w_1$  and  $w_2$  such that  $l(\gamma) \leq M\|w_1 - w_2\|$ . Also, we let  $\Gamma = H(\gamma)$ . On one hand, by (2.3), (2.4) and Lemma A, we find that

$$\begin{aligned} l(\Gamma) &= \int_{\Gamma} \|dH(w)\| \leq \int_{\gamma} (\|DH(w)\| + \|\overline{DH}(w)\|) \|dw\| \\ &\leq \int_{\gamma} (\|I_n\| + 2\|Dg[Dh]^{-1} (I_n - Dg[Dh]^{-1})^{-1}\|) \|dw\| \\ &\leq \int_{\gamma} \left(1 + \frac{2\|Dg[Dh]^{-1}\|}{1 - \|Dg[Dh]^{-1}\|}\right) \|dw\| \\ (2.5) \quad &\leq \frac{1+C}{1-C} M \|w_2 - w_1\|. \end{aligned}$$

On the other hand, the definition of  $H$  gives

$$\begin{aligned} (2.6) \quad \|H(w_2) - H(w_1)\| &\geq \|w_2 - w_1\| - 2\|g(F^{-1}(w_2)) - g(F^{-1}(w_1))\| \\ &\geq \|w_2 - w_1\| - 2 \int_{\gamma} \|Dg[DF]^{-1}\| \|dw\| \\ &\geq \|w_2 - w_1\| - 2 \int_{\gamma} \frac{\|Dg[Dh]^{-1}\|}{1 - \|Dg[Dh]^{-1}\|} \|dw\| \\ &\geq \frac{1 - C(1 + 2M)}{1 - C} \|w_2 - w_1\|, \end{aligned}$$

and therefore, (2.5) gives

$$l(\Gamma) \leq M' \|H(w_2) - H(w_1)\|,$$

where  $M' = \frac{(1+C)M}{1-(1+2M)C}$ .

Finally, we show that  $f_A = h + \bar{g}A$  is univalent for every  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $\|A\| \leq 1$ . Suppose on the contrary that  $f_A$  fails to be univalent. Then there are two distinct points  $w_1, w_2$  such that  $H(w_1) = H(w_2)$  which is impossible, by (2.6).

At last, since  $f_A = h + \bar{g}A$  is univalent for every  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $\|A\| \leq 1$ , it follows from Lemma 1 that  $h + g$  belongs to SBH and the proof of part (I) is finished.

Now we begin to prove part (II). Assume the hypotheses and Define

$$H(w) = F_A(f^{-1}(w)) = w - \overline{G(w)} - G(w)A,$$



where  $G = g \circ f^{-1}$  and  $w = f(z)$ . By the chain rule, we have

$$DG = DgDf^{-1} \quad \text{and} \quad \overline{DG} = Dg\overline{Df^{-1}},$$

which implies

$$DH = I_n - \overline{Dg} \overline{Df^{-1}} - DgDf^{-1}A \quad \text{and} \quad \overline{DH} = -\overline{Dg} \overline{Df^{-1}} - Dg\overline{Df^{-1}}A.$$

It follows from the inverse mapping theorem and Lemma A that  $f^{-1}$  is differentiable. Differentiation of the equation  $f^{-1}(f(z)) = z$  yields the following relations

$$\begin{cases} Df^{-1}Dh + \overline{Df^{-1}}Dg = I_n, \\ Df^{-1}\overline{Dg} + \overline{Df^{-1}}\overline{Dh} = 0, \end{cases}$$

which give

$$(2.7) \quad \begin{cases} Df^{-1} = [Dh]^{-1} (I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1})^{-1}, \\ \overline{Df^{-1}} = -[Dh]^{-1} (I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1})^{-1} \overline{Dg}[\overline{Dh}]^{-1}. \end{cases}$$

By (2.7) and Lemma A, we get

$$\begin{aligned} \|DG\| + \|\overline{DG}\| &= \|DgDf^{-1}\| + \|Dg\overline{Df^{-1}}\| \\ &= \|Dg[Dh]^{-1} (I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1})^{-1}\| \\ &\quad + \|Dg[Dh]^{-1} (I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1})^{-1} \overline{Dg}[\overline{Dh}]^{-1}\| \\ &\leq \| (I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1})^{-1} \| \|Dg[Dh]^{-1}\| \\ &\quad \times (1 + \|Dg[Dh]^{-1}\|) \\ &\leq \frac{\|Dg[Dh]^{-1}\| (1 + \|Dg[Dh]^{-1}\|)}{1 - \|\overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\|} \\ &\leq \frac{\|Dg[Dh]^{-1}\| (1 + \|Dg[Dh]^{-1}\|)}{1 - \|Dg[Dh]^{-1}\|^2} \\ &\leq \frac{\|Dg[Dh]^{-1}\|}{1 - \|Dg[Dh]^{-1}\|} \\ &< \frac{C}{1 - C}. \end{aligned}$$

Let  $\gamma \subset \Omega$  be a curve joining  $w_1, w_2$  with  $l(\gamma) \leq M\|w_1 - w_2\|$ .

Then

$$\begin{aligned} \|DH\| + \|\overline{DH}\| &= \|I_n - \overline{Dg} \overline{Df^{-1}} - DgDf^{-1}A\| + \|\overline{Dg} \overline{Df^{-1}} + Dg\overline{Df^{-1}}A\| \\ &\leq 1 + 2\|\overline{Dg} \overline{Df^{-1}}\| + 2\|\overline{Dg} \overline{Df^{-1}}\| \\ &\leq 1 + \frac{2\|Dg[Dh]^{-1}\|}{1 - \|Dg[Dh]^{-1}\|} \\ &= \frac{1 + \|Dg[Dh]^{-1}\|}{1 - \|Dg[Dh]^{-1}\|} \\ &\leq \frac{1 + C}{1 - C}, \end{aligned}$$

which gives

$$\begin{aligned}
 l(H(\gamma)) &\leq \int_{\gamma} (\|DH(w)\| + \|\overline{D}H(w)\|) \|dw\| \\
 &\leq \frac{1+C}{1-C} l(\gamma) \\
 (2.8) \quad &\leq \frac{M(1+C)}{1-C} \|w_2 - w_1\|.
 \end{aligned}$$

On the other hand, by (2.8),

$$\begin{aligned}
 \|H(w_2) - H(w_1)\| &= \|w_2 - \overline{G(w_2)} - G(w_2)A - (w_1 - \overline{G(w_1)} - G(w_1)A)\| \\
 &\geq \|w_2 - w_1\| - 2\|G(w_2) - G(w_1)\| \\
 &\geq \|w_2 - w_1\| - 2 \int_{\gamma} (\|DG(w)\| + \|\overline{D}G(w)\|) \|dw\| \\
 &\geq \|w_2 - w_1\| - \frac{2C}{1-C} l(\gamma) \\
 (2.9) \quad &\geq \frac{1 - C(1+2M)}{1-C} \|w_2 - w_1\| \\
 &\geq [1 - C(1+2M)] \frac{l(H(\gamma))}{M(1+C)}.
 \end{aligned}$$

Hence

$$l(H(\gamma)) \leq \frac{M(1+C)}{1-C(1+2M)} \|H(w_2) - H(w_1)\|.$$

Now we prove the univalence of  $F_A$  for every  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $\|A\| \leq 1$ . Suppose that  $H(w_1) = H(w_2)$  for distinct points  $w_1, w_2 \in f(\mathbb{B}^n)$ . Then there exist two distinct points  $w_1, w_2$  such that  $H(w_1) = H(w_2)$  which is a contradiction to (2.9).

At last, since  $F_A$  is biholomorphic for every  $A \in L(\mathbb{C}^n, \mathbb{C}^n)$  with  $\|A\| \leq 1$ , it follows Lemma 1 that  $h + \overline{g}$  is in SPU. The proof of the theorem is complete.  $\square$

### 3. THE LANDAU-BLOCH THEOREM ON PLURIHARMONIC MAPPINGS

For  $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$ , we use the standard notations:

$$(3.1) \quad \Lambda_f(z) = \max_{\theta \in \partial \mathbb{B}^n} \|Df(z)\theta + \overline{D}f(z)\overline{\theta}\| \quad \text{and} \quad \lambda_f(z) = \min_{\theta \in \partial \mathbb{B}^n} \|Df(z)\theta + \overline{D}f(z)\overline{\theta}\|.$$

We see that (see for instance [4, 8])

$$(3.2) \quad \Lambda_f = \max_{\theta \in \partial \mathbb{B}_{\mathbb{R}}^{2n}} \|J_f \theta\| \quad \text{and} \quad \lambda_f = \min_{\theta \in \partial \mathbb{B}_{\mathbb{R}}^{2n}} \|J_f \theta\|.$$

Then the following two results are useful for the proof of Theorem 3.

**Theorem B.** ([4, Theorem 4]) *Let  $f$  be a pluriharmonic mappings of  $\mathbb{B}^n$  into  $\mathbb{B}^m$ . Then*

$$\Lambda_f(z) \leq \frac{4}{\pi} \frac{1}{1 - \|z\|^2} \quad \text{for } z \in \mathbb{B}^n.$$

*If  $f(0) = 0$ , then  $\|f(z)\| \leq (4/\pi) \arctan \|z\|$  for  $z \in \mathbb{B}^n$ .*

**Lemma C.** ([4, Lemma 1] or [17, Lemma 4]) *Let  $A$  be an  $n \times n$  complex (real) matrix with  $\|A\| \neq 0$ . Then for any unit vector  $\theta \in \partial\mathbb{B}^n$ , the inequality  $\|A\theta\| \geq |\det A|/\|A\|^{n-1}$  holds.*

**Proof of Theorem 3.** For each fixed  $\theta \in \partial\mathbb{B}^n$ , let  $A_\theta = Dg[Dh]^{-1}\theta$ . Since  $\|Dg[Dh]^{-1}\| < 1$ , by Schwarz's lemma, we see that for  $z \in \mathbb{B}^n(r)$ ,  $\|A_\theta(z)\| < \|z\|$  if  $r \in (0, 1)$ . The arbitrariness of  $\theta \in \partial\mathbb{B}^n$  gives

$$(3.3) \quad \|Dg(z)[Dh(z)]^{-1}\| < r$$

for  $z \in \mathbb{B}^n(r)$ .

By (3.3), we get

$$(3.4) \quad \frac{1 + \|Dg(z)[Dh(z)]^{-1}\|}{1 - \|Dg(z)[Dh(z)]^{-1}\|} \leq \frac{1 + r}{1 - r} = K_r, \quad z \in \mathbb{B}^n(r).$$

Applying (3.4), we obtain that

$$\begin{aligned} V_f(r) &= \int_{\mathbb{B}^n(r)} \|Dh(\zeta)\|^{2n} (1 - \|Dg(z)[Dh(z)]^{-1}\|^2)^n dV(\zeta) \\ &\geq \frac{1}{K_r^n} \int_{\mathbb{B}^n(r)} \|Dh(\zeta)\|^{2n} (1 + \|Dg(z)[Dh(z)]^{-1}\|)^{2n} dV(\zeta). \end{aligned}$$

Fix  $z \in \mathbb{B}^n$  and let  $D_z^r = \{\zeta \in \mathbb{C}^n : \|\zeta - z\| < r - \|z\|\}$ . Since  $\|Dh\theta + \overline{Dg}\bar{\theta}\|^{2n}$  is subharmonic in  $\mathbb{B}^n$ , we see that for  $\rho \in [0, 1 - \|z\|]$ ,

$$(3.5) \quad \|Dh(z)\theta + \overline{Dg(z)}\bar{\theta}\|^{2n} \leq \int_{\partial\mathbb{B}^n} \|Dh(z + \rho\zeta)\theta + \overline{Dg(z + \rho\zeta)}\bar{\theta}\|^{2n} d\sigma(\zeta),$$

where  $\theta \in \partial\mathbb{B}^n$ . Multiplying both sides of (3.5) by  $2nr^{2n-1}$  and then integrating from 0 to  $r - \|z\|$ , we obtain

$$\begin{aligned} &(r - \|z\|)^{2n} \|Dh(z)\theta + \overline{Dg(z)}\bar{\theta}\|^{2n} \\ &\leq \int_0^{r-\|z\|} \left[ 2n\rho^{2n-1} \int_{\partial\mathbb{B}^n} \|Dh(z + \rho\zeta)\theta + \overline{Dg(z + \rho\zeta)}\bar{\theta}\|^{2n} d\sigma(\zeta) \right] d\rho \\ &= \int_{D_z^r} \|Dh(\zeta)\theta + \overline{Dg(\zeta)}\bar{\theta}\|^{2n} dV(\zeta) \\ &\leq \int_{\mathbb{B}^n(r)} \|Dh(\zeta)\theta + \overline{Dg(\zeta)}\bar{\theta}\|^{2n} dV(\zeta) \\ &\leq \int_{\mathbb{B}^n(r)} (\|Dh(\zeta)\| + \|\overline{Dg(\zeta)}\|)^{2n} dV(\zeta) \\ &\leq \int_{\mathbb{B}^n(r)} (1 + \|Dg(\zeta)[Dh(\zeta)]^{-1}\|)^{2n} \|Dh(\zeta)\|^{2n} dV(\zeta) \\ &\leq K_r^n V_f(r) \\ &\leq K_r^n V, \end{aligned}$$

which implies that

$$(3.6) \quad \Lambda_f^{2n}(z) = \max_{\theta \in \partial \mathbb{B}^n} \|Dh(z)\theta + \overline{Dg(z)}\bar{\theta}\|^{2n} \leq \frac{K_r^n V}{(r - |z|)^{2n}}.$$

For  $\xi \in \mathbb{B}^n$ , let  $F(\xi) = r^{-1}f(r\xi) = H(\xi) + \overline{G(\xi)}$ , where

$$H(\xi) = r^{-1}h(r\xi) \quad \text{and} \quad G(\xi) = r^{-1}g(r\xi).$$

Then

$$(3.7) \quad \Lambda_F(\xi) = \max_{\theta \in \partial \mathbb{B}^n} \|DH(\xi)\theta + \overline{DG(\xi)}\bar{\theta}\| \leq \frac{K_r^{\frac{1}{2}} V^{\frac{1}{2n}}}{r(1 - |\xi|)},$$

where  $z = r\xi$ . By (3.7), we have

$$(3.8) \quad \Lambda_F(\xi) \leq \frac{V^{\frac{1}{2n}} \sqrt{\min_{0 < r < 1} \psi(r)}}{1 - |\xi|} = \frac{V^{\frac{1}{2n}} \sqrt{\psi(r_0)}}{1 - |\xi|},$$

where

$$\psi(r) = \frac{1+r}{r^2(1-r)} \quad \text{and} \quad r_0 = \frac{\sqrt{5}-1}{2}.$$

Again, for  $w \in \mathbb{B}^n$  and a fixed  $t \in (0, 1)$ , let  $P(w) = t^{-1}F(tw)$ . Then

$$(3.9) \quad \Lambda_P(w) = \Lambda_F(\xi) \leq \frac{V^{\frac{1}{2n}} \sqrt{\psi(r_0)}}{1 - |\xi|} = \frac{V^{\frac{1}{2n}} \sqrt{\psi_0}}{1 - t|w|} \leq \frac{V^{\frac{1}{2n}} \sqrt{\psi_0}}{1 - t} = M(t)$$

where  $\psi_0 = \psi(r_0)$  and by a computation, it follows easily that

$$\psi_0 = \frac{11 + 5\sqrt{5}}{2} \approx 11.09017.$$

Let  $w_1, w_2$  be two distinct points in  $\mathbb{B}^n(\rho(t))$  with  $w_2 - w_1 = \|w_2 - w_1\|\theta$ , where

$$\rho(t) = \frac{\alpha\pi}{4(M(t) + M(0))(M(0))^{2n-1}} = \frac{\alpha\pi(1-t)}{4(M(0))^{2n}(2-t)} = \frac{\alpha\pi(1-t)}{4V\psi_0^{2n}(2-t)}.$$

By the assumption and monotonicity of  $\rho(t)$ , it can be easily see that for  $\rho(t) \leq \rho(0) \leq 1$  for  $t \in [0, 1)$ . Define the pluriharmonic mapping

$$\phi_\theta(w) = (DP(w) - DP(0))\theta + (\overline{DP}(w) - \overline{DP}(0))\bar{\theta} \quad \text{for } w \in \mathbb{B}^n.$$

By (3.9), we get

$$\|\phi_\theta(w)\| \leq \Lambda_P(w) + \Lambda_P(0) \leq M(t) + M(0)$$

and therefore, by using Theorem B, we obtain

$$(3.10) \quad \|\phi_\theta(w)\| \leq \frac{4(M(t) + M(0))}{\pi} \arctan \|w\| \leq \frac{4(M(t) + M(0))}{\pi} \|w\|$$

for  $w \in \mathbb{B}^n$ . By (3.1), (3.2), (3.8) and Lemma C, we have

$$(3.11) \quad \lambda_P(0) \geq \frac{|\det J_P(0)|}{\Lambda_P^{2n-1}(0)} \geq \frac{\alpha}{(M(0))^{2n-1}}.$$

Let  $[w_1, w_2]$  denote the segment from  $w_1$  to  $w_2$ ,  $dw = (du_1, \dots, du_n)^T$  and  $d\bar{w} = (d\bar{u}_1, \dots, d\bar{u}_n)^T$ . Then by (3.10) and (3.11) we have

$$\begin{aligned}
\|P(w_1) - P(w_2)\| &= \left\| \int_{[w_1, w_2]} DP(w) dw + \overline{DP}(w) d\bar{w} \right\| \\
&\geq \left\| \int_{[w_1, w_2]} DP(0) dw + \overline{DP}(0) d\bar{w} \right\| \\
&\quad - \left\| \int_{[w_1, w_2]} (DP(w) - DP(0)) dw + (\overline{DP}(w) - \overline{DP}(0)) d\bar{w} \right\| \\
&\geq \|w_1 - w_2\| \lambda_P(0) - \int_{[w_1, w_2]} \|\phi_\theta(w)\| \|dw\| \\
&> \|w_1 - w_2\| \left[ \frac{\alpha}{(M(0))^{2n-1}} - \frac{4(M(t) + M(0))}{\pi} \rho(t) \right] = 0.
\end{aligned}$$

Furthermore, for  $w$  with  $|w| = \rho(t)$ , we have

$$\begin{aligned}
\|P(w) - P(0)\| &\geq \left\| \int_{[0, w]} DP(0) d\zeta + \overline{DP}(0) d\bar{\zeta} \right\| \\
&\quad - \left\| \int_{[0, w]} (DP(\zeta) - DP(0)) d\zeta + (\overline{DP}(\zeta) - \overline{DP}(0)) d\bar{\zeta} \right\| \\
&> \rho(t) \left[ \frac{\alpha}{(M(0))^{2n-1}} - \frac{2(M(t) + M(0))}{\pi} \rho(t) \right] \\
&= \frac{\alpha^2 \pi}{8(M(t) + M(0))(M(0))^{4n-2}} \\
&= \frac{\alpha^2 \pi (1-t)}{8(M(0))^{4n-1} (2-t)}.
\end{aligned}$$

The last inequality shows that the range  $P(\mathbb{B}^n(\rho(t)))$  contains a univalent ball  $\mathbb{B}^n(R)$ , where

$$R = R(t) \geq \frac{\alpha^2 \pi (1-t)}{8(M(0))^{4n-1} (2-t)}.$$

Therefore,  $f$  is univalent in  $\mathbb{B}^n(tr_0 \rho(t))$ , and  $f(\mathbb{B}^n(tr_0 \rho(t)))$  covers the ball  $\mathbb{B}^n(R_c)$  with

$$\begin{aligned}
R_c = \max_{0 < t < 1} (r_0 t R(t)) &\geq \frac{\alpha^2 \pi r_0 \max_{0 < t < 1} \nu(t)}{8(M(0))^{4n-1}} \\
&= \frac{\alpha^2 \pi r_0 (3 - 2\sqrt{2})}{8(M(0))^{4n-1}} \\
&= \frac{\alpha^2 \pi (\sqrt{5} - 1)(3 - 2\sqrt{2})}{16V^{\frac{4n-1}{2n}} \psi_0^{\frac{4n-1}{2}}},
\end{aligned}$$

where  $\nu(t) = \frac{t(1-t)}{(2-t)}$  and a computation gives that

$$\max_{0 < t < 1} \nu(t) = \nu(2 - \sqrt{2}) = 3 - 2\sqrt{2}.$$

Moreover, using the value of  $r_0$  and the function  $\rho(t)$ , we find that

$$tr_0 \rho(t) = \frac{\alpha \pi (\sqrt{5} - 1)}{8V\psi_0^n} \nu(t) \geq \frac{\alpha \pi (\sqrt{5} - 1)(3 - 2\sqrt{2})}{8V\psi_0^n} = R_u.$$

The proof of this theorem is complete.  $\square$

**Proof of Theorem 4.** Applying Lemma C, for an  $n \times n$  complex matrix  $A$ , we have

$$(3.12) \quad |\det A| \leq \|A\|^n.$$

By (3.4) and (3.12), for  $z \in \mathbb{B}^n(r)$ , we get

$$\begin{aligned} |\det J_f(z)| &= |\det Dh(z)|^2 \left| \det (I_n - Dg(z)[Dh(z)]^{-1} \overline{Dg(z)[Dh(z)]^{-1}}) \right| \\ &\leq \|Dh(z)\|^{2n} \left\| (I_n - Dg(z)[Dh(z)]^{-1} \overline{Dg(z)[Dh(z)]^{-1}}) \right\|^n \\ &\leq \|Dh(z)\|^{2n} (1 + \|Dg(z)[Dh(z)]^{-1}\|^2)^n \\ &\leq \|Dh(z)\|^{2n} (1 + \|Dg(z)[Dh(z)]^{-1}\|)^{2n} \\ &\leq \left( \frac{1+r}{1-r} \right)^n \|Dh(z)\|^{2n} (1 - \|Dg(z)[Dh(z)]^{-1}\|^2)^n, \end{aligned}$$

which implies that

$$\begin{aligned} \int_{\mathbb{B}^n(r)} |\det J_f(z)| dV(z) &\leq K_r^n \int_{\mathbb{B}^n(r)} \|Dh(z)\|^{2n} (1 - \|Dg(z)[Dh(z)]^{-1}\|^2)^n dV(z) \\ &= K_r^n V_f(r), \end{aligned}$$

where  $K_r = \frac{1+r}{1-r}$ . The proof of this theorem is complete.  $\square$

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